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D-branes, discrete torsion and the McKay correspondence

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Abstract: We analyze the D-branes of a type-IIB string theory on an orbifold singularity including the possibility of discrete torsion following the work of Douglas et al. First we prove some general results about the moduli space of a point associated to the "regular representation" of the orbifold group. This includes some analysis of the "wrapped branes" which necessarily appear when the orbifold singularity is not isolated. Next we analyze the stringy homology of the orbifold using the McKay correspondence and the relationship between K-theory and homology. We find that discrete torsion and torsion in this stringy homology are closely-related concepts but that they differ in general. Lastly we question to what extent the D-1 brane may be thought of as being dual to a string.

Keywords: Superstrings and Heterotic Strings, D-branes, Superstring Vacua.

JHEP02(2001)009

Contents

1. Introduction **1**

2. The Γ-equivariant D-brane quiver **3**

3. D-brane moduli space **7**
- 3.1 The regular representation 7
- 3.2 Mobile wrapped branes 10

4. The McKay correspondence **10**
- 4.1 An attempt with homology 10
- 4.2 K-theory 13

5. Examples **16**
- 5.1 $\mathbb{Z}_2 \times \mathbb{Z}_2$ 16
  - 5.1.1 No discrete torsion 16
  - 5.1.2 With discrete torsion 18
- 5.2 A trihedral group 19
  - 5.2.1 No discrete torsion 20
  - 5.2.2 With discrete torsion 21

6. Discussion of strings versus D-branes **21**

1. Introduction

D-branes as probes of string theory in nontrivial backgrounds have been a source of useful insights since [1]. In [2] Douglas and Moore used a quotient construction to study the dynamics of branes at an orbifold singularity. The construction involves a choice of a representation of the quotient group on the Chan-Paton indices. This leads to a supersymmetric gauge theory on the worldvolume (as we review later on). The moduli space of vacua of this theory contains an approximation to the geometry of the transverse space in which the brane is free to move. The parameters in the theory are determined by the closed-string background; the parameter space thus probes the moduli space of closed-string vacua. We refer to [3] and references therein for a larger account of this work.

Let us consider a type-IIB string on the local form of an orbifold $\mathbb{C}^n/\Gamma$ for some finite group $\Gamma \subset \mathrm{SU}(n)$. It has been clear from the early days of this subject that D-branes give a particularly direct physical picture of the "McKay Correspondence".

JHEP02(2001)009

That is, there is a relationship between the even-dimensional homology, or more properly K-theory, of the resolution of such an orbifold and the representation theory of Γ. (We refer to [4] for a review of the mathematics of the McKay correspondence.) It was realized in [5] that by computing the masses of wrapped branes as one blew-up a singularity, one could relate the homology of the blow-up to representations of Γ. This correspondence was made much more explicit in [6] where the explicit map between homology and representation theory was computed.

Douglas [7] was able to extend the work of [2] by considering *projective* representations. It is very natural to associate this degree of freedom with Vafa's discrete torsion [8]. It would be nice to extend the McKay correspondence to include discrete torsion. The work of Gomis [9] is related to this question and we explore the subject further in this paper. There has also been some work associating discrete torsion with monodromy acting on partial resolutions [10, 11, 12]. We will not discuss this here. Note also that we will completely ignore anything to do with odd-dimensional cycles and deformations of complex structure. We are only interested in data associated to the complexified Kähler form.

Our aim in this paper is to cover the following topics:

1. We will present a computation of the moduli space of vacua determined by the "regular representation" and its relation to the target space geometry in the general case. This involves extending some proofs by Sardo Infirri [13, 14] to the case of discrete torsion. We will also discuss how wrapped branes appear when the singularity is not isolated. Much has already been said about these topics in the context of specific examples and various generalizations. See, for example, [15, 6, 16, 17].

2. We will use the McKay correspondence for K-theory to *define* a theory of stringy homology on the orbifold itself. In the absence of discrete torsion this agrees with the known results obtained by studying the masses of wrapped branes. The stringy homology will exhibit torsion when, and only when, discrete torsion is switched on. This construction will be closely related to observations in [9].

3. We will discuss if there is such a thing as an S-duality of the type-IIB string which literally exchanges the string and the D1-brane.

Of particular note in the second case is the relationship between torsion in homology and discrete torsion. This has been analyzed in some cases (see, for example, [18]) but the general picture remains unclear. At least for the local analysis of an orbifold we hope to shed some light on this question. We will see that the torsion in homology naturally contains a subgroup of $\mathbb{Z}_p$, where p is the order of the element of discrete torsion within the group $H^2(\Gamma, \mathrm{U}(1))$.

In section 2 we set up many aspects of the group theory we use and describe how to construct quiver diagrams. In section 3 we give a general proof that the moduli space of the D-brane associated with a point really is the target space $\mathbb{C}^n/\Gamma$ only in the case of an isolated singularity. We give the general description of the non-isolated case in terms of the familiar "wrapped branes" which are associated to induced representations of Γ.

In section 4 we set up the McKay correspondence for orbifolds with discrete torsion. This amounts to defining what we mean by homology.

The previous sections are clarified by reviewing some examples in section 5. We draw the quiver corresponding to the field theory, analyze the possibilities for wrapped branes and compute some stringy homology groups.

Finally in section 6 we emphasize the relationship between torsion in homology and discrete torsion. We also discuss the S-duality of the type-IIB string.

2. The Γ-equivariant D-brane quiver

In this section we will encode the data associated to a D-brane on an orbifold in terms of a quiver diagram. We will be general here and present the analysis for any representation of the orbifold group and any choice of discrete torsion. For definiteness, we will discuss the case mentioned above, of D3-branes near a singularity of the local form $\mathbb{C}^3/\Gamma$. The worldvolume theory is then an $\mathcal{N}=1$ supersymmetric theory in four dimensions. Our analysis follows the discussion in [14] with the (small) modifications required to incorporate discrete torsion.

The orbifold theory is constructed following [2] as a quotient from a theory of branes on the covering space. Let the brane be located at a point in $\mathbb{C}^n$ which is fixed under all of Γ. We will assume this point is the origin. Let R be an m-dimensional representation of Γ. We then consider m branes on the covering space. The low-energy dynamics on the brane worldvolume is thus a $\mathrm{U}(m)$ gauge theory[1] with $\mathcal{N}=4$ supersymmetry in four dimensions. In terms of the $\mathcal{N}=1$ supersymmetry that is unbroken by the quotient, this is a gauge theory with three chiral multiplets X_i in the adjoint representation (corresponding to the three complex transverse coordinates) and a superpotential given by

$$W = \mathrm{tr}\left(X_1[X_2, X_3]\right). \tag{2.1}$$

The representation R defines a *lift* of the action of the orbifolding group on $\mathbb{C}^n$ to the gauge group. The orbifold theory is obtained [2] by projecting onto the invariant degrees of freedom under the combined action of Γ. The theory we obtain under

[1]The diagonal $\mathrm{U}(1)$ always decouples in the gauge theory so the effective gauge group is only really $\mathrm{SU}(m)$.

the quotienting process depends very much on R. To begin with, and to fix our notations, we take R to be a linear[2] representation, corresponding to the absence of discrete torsion.

The quotient theory will also be a gauge theory. Because the gauge fields live in the directions tangent to the brane, Γ acts on these only via R. The gauge fields in the quotient theories will generate the subgroup of $\mathscr{G} = \mathrm{U}(m)$ which is invariant under this action

$$R(\gamma)AR(\gamma)^{-1} = A\,, \tag{2.2}$$

for all $\gamma \in \Gamma$. To facilitate the following discussions we will introduce some fancier notation. First let us complexify the gauge group. Our initial complexified gauge group will be $\mathscr{G}_{\mathbb{C}} = \mathrm{GL}(m,\mathbb{C}) = \mathrm{End}(\mathbb{C}^m)$. The invariant complexified gauge group $\mathscr{G}_{\mathbb{C}}^{\Gamma}$ is then generated by all matrices A which satisfy (2.2). We write this as

$$\mathscr{G}_{\mathbb{C}}^{\Gamma} = (\mathrm{End}\, R)^{\Gamma}\,. \tag{2.3}$$

These gauge fields will appear in appropriate supermultiplets. In our case we find vector multiplets.

Additional degrees of freedom in the low-energy theory come from the invariant degrees of freedom contained in the chiral multiplets. Γ acts on these, in addition to its embedding in $\mathscr{G}$, via the n-dimensional representation Q determining its action on $\mathbb{C}^n$. The invariant degrees of freedom now satisfy

$$\sum_{j=1}^{n} Q_{ij}(\gamma)R(\gamma)X_jR(\gamma)^{-1} = X_i\,, \tag{2.4}$$

for all $\gamma \in \Gamma$. In our more abstract notation this is written as

$$X \in (Q \otimes \mathrm{End}\, R)^{\Gamma}\,. \tag{2.5}$$

In addition to the gauge couplings, these interact via a cubic superpotential that is simply the restriction to the invariant fields of the superpotential (2.1).

To allow for discrete torsion we allow R to be a *projective* representation. That is

$$R(\gamma)R(\mu) = \alpha(\gamma,\mu)R(\gamma\mu)\,, \tag{2.6}$$

where $\alpha(\gamma,\mu) \in \mathrm{U}(1)$.[3] The associativity of group multiplication is consistent with this provided

$$\alpha(\gamma,\mu)\alpha(\gamma\mu,\rho) = \alpha(\gamma,\mu\rho)\alpha(\mu,\rho)\,, \tag{2.7}$$

which is precisely the condition that $\alpha(\gamma,\mu)$ is a group 2-cocycle. Further, it is clear that two cocycles related by

$$\alpha'(\gamma,\mu) = \frac{\beta(\gamma)\beta(\mu)}{\beta(\gamma\mu)}\alpha(\gamma,\mu)\,, \tag{2.8}$$

for any map $\beta : \Gamma \to \mathrm{U}(1)$ lead to equivalent conditions (2.6). This shows that the

[2] Non-projective.

[3] This implies that $\alpha(1,\gamma) = \alpha(\gamma,1) = 1$ for any γ.

JHEP02(2001)009

allowed representations are determined by α as an element of $H^2(\Gamma, \mathrm{U}(1))$. A choice of a cohomology class α then determines the phases $\epsilon(\gamma,\mu)$ associated to twisted sectors in Vafa's formulation [8] by $\epsilon(\gamma,\mu) = \alpha(\gamma,\mu)\alpha(\mu,\gamma)^{-1}$ (note this need only be defined for commuting γ, μ). We refer to [7, 9] for more details.

We can now repeat the calculation of the quotient theory as above. We find it useful to resort to fairly algebraic language. Any standard text on the representation theory of finite groups should explain all the terms we use.

Let us introduce the $\mathbb{C}$-algebra $\mathbb{C}^\alpha\Gamma$ defined as follows. Firstly we let e_γ, $\gamma \in \Gamma$, be a basis for $\mathbb{C}^\alpha\Gamma$. That is, any element may be written uniquely as a sum

$$\sum_{\gamma\in\Gamma} c_\gamma e_\gamma \,, \tag{2.9}$$

where $c_\gamma \in \mathbb{C}$. We define a distributive product by

$$e_\gamma e_\mu = \alpha(\gamma,\mu) e_{\gamma\mu} \,. \tag{2.10}$$

The standard theory of representations and modules now says that *any projective representation of* Γ *can be written as a* $\mathbb{C}^\alpha\Gamma$*-module.* Naturally we may obtain linear representations by using the trivial α to obtain $\mathbb{C}\Gamma$-modules.

Let R_l represent the *irreducible* representations of Γ twisted by $\alpha \in H^2(\Gamma, \mathrm{U}(1))$. It is well-known that irreducible *linear* representations of Γ are counted by the number of conjugacy classes of Γ. There is a similar concept for projective representations. An "α-regular" conjugacy class contains elements γ satisfying $\alpha(\gamma,\mu) = \alpha(\mu,\gamma)$ for all $\mu \in \Gamma$ such that $\gamma\mu = \mu\gamma$. The number of irreducible projective representations is equal to the number of α-regular conjugacy classes.

We may view R_l as *simple* $\mathbb{C}^\alpha\Gamma$-modules. We refer to [19] for an exposition of the properties of irreducible projective representations. We may then decompose

$$R = \bigoplus_l V_l \otimes R_l \,, \tag{2.11}$$

where V_l is a linear vector space whose dimension is determined by the number of times a given irreducible representation appears in R.

Let us use Hom_Γ to denote the group of homomorphisms of $\mathbb{C}^\alpha\Gamma$-modules. We may now deduce that for $A \in \mathscr{G}^\Gamma_{\mathbb{C}}$

$$\begin{aligned} A &\in (\mathrm{End}\, R)^\Gamma \\ &= \mathrm{Hom}_\Gamma(R,R) \\ &= \mathrm{Hom}_\Gamma\left(\bigoplus_l R_l \otimes V_l, \bigoplus_m R_m \otimes V_m\right) \\ &= \bigoplus_l \mathrm{Hom}(V_l, V_l) \\ &= \bigoplus_l \mathrm{End}(V_l)\,, \end{aligned} \tag{2.12}$$

where the penultimate step was obtained by using Schur's Lemma. Thus if we denote $v_l = \dim(V_l)$ we see that $\mathscr{G}_{\mathbb{C}}^{\Gamma} = \prod_l \mathrm{GL}(v_l)$. In other words, restricting back to the compact real form, we have a gauge group

$$\mathscr{G}^{\Gamma} = \prod_l \mathrm{U}(v_l)\,. \tag{2.13}$$

Since Q is a linear representation of Γ and R is a projective representation twisted by α, it is not hard to show that $Q \otimes R$ is a projective representation twisted by α. We may thus decompose

$$Q \otimes R_l = \bigoplus_m A_{lm} \otimes R_m\,, \tag{2.14}$$

where A_{lm} are vector spaces.

We may then deduce

$$\begin{aligned} X \in\ & (Q \otimes \mathrm{End}\, R)^{\Gamma} \\ &= \mathrm{Hom}_{\Gamma}(R, Q \otimes R) \\ &= \mathrm{Hom}_{\Gamma}\left(\bigoplus_l R_l \otimes V_l, Q \otimes \left(\bigoplus_m R_m \otimes V_m\right)\right) \\ &= \mathrm{Hom}_{\Gamma}\left(\bigoplus_l R_l \otimes V_l, \bigoplus_{mn} A_{mn} \otimes R_n \otimes V_m\right), \\ &= \bigoplus_{lm} A_{lm} \otimes \mathrm{Hom}(V_l, V_m)\,. \end{aligned} \tag{2.15}$$

As above, in addition to the gauge couplings, the chiral multiplets interact via a cubic superpotential descended from (2.1).

A useful way of visualizing the structure of the resulting theory is in terms of a *quiver*.

1. There is a node for every irreducible representation R_l. Each node is labelled by $v_l = \dim V_l$. Each node represents a factor of $\mathrm{U}(v_l)$ in the gauge group $\mathscr{G}^{\Gamma}$.

2. From node l to node m we draw $a_{lm} = \dim A_{lm}$ arrows. Each arrow represents a contribution of $\mathrm{Hom}(V_l, V_m)$ to the space in which X lives. That is, the arrow represents a $(\overline{\mathbf{v}}_l, \mathbf{v}_m)$ representation of $\mathrm{U}(v_l) \times \mathrm{U}(v_m)$.

Note that the numbers a_{lm}, and hence the arrows, are fixed by Γ and α and do not depend on R. The numbers v_l do depend on R. If $v_l = 0$ for a particular node then one can ignore that vertex and the associated arrows. We will give several examples of quivers below.

3. D-brane moduli space

One object of immediate interest is the moduli space of classical vacua of the gauge theory. The is the zero string-coupling limit of the moduli space of the associated brane in the orbifold background. The classical vacua are parameterized by values for the scalars in the chiral multiplets, modulo gauge equivalence, solving F-term equations (critical points of the superpotential) as well as the D-term equations (zero moment maps for the action of $\mathscr{G}^\Gamma$). The details of these depend upon R, but since the quotient theory is obtained from the covering theory simply be reducing the gauge group to a subgroup of $\mathscr{G}$ and setting some of the chiral fields to zero, we can give a uniform description directly in terms of the covering theory. From (2.1) we have the F-terms

$$[X_i, X_j] = 0\,, \qquad \text{for all } i, j \tag{3.1}$$

where in a particular quotient we keep only the invariant components of X. The moment map for $\mathscr{G}$ is simply

$$\sum_{i=1}^{n} [X_i^\dagger, X_i] = 0\,. \tag{3.2}$$

In the quotient theory, only the parts of this corresponding to $\mathscr{G}^\Gamma$ will be nontrivial. One then divides the solution set of these equations by the gauge group $\mathscr{G}^\Gamma$ to obtain $\mathscr{M}_R$. The key idea of the physics of D-branes on the orbifold is the following

Proposition 1. *The space $\mathscr{M}_R$ represents the space of allowed positions of the D-brane associated to R in the target space $\mathbb{C}^n/\Gamma$.*

The simplest case of using an irreducible representation for R tends to give $\mathscr{M}_R$ equal to a point — the wrapped D-brane is stuck at the origin. One obtains more interesting results by using bigger representations.

3.1 The regular representation

Of particular note is the case when R is given by putting $R = \mathbb{C}^\alpha\Gamma$. (That is, we view $R = \mathbb{C}^\alpha\Gamma$ itself as a $\mathbb{C}^\alpha\Gamma$-module.) When α is trivial, this is equivalent to making R the *regular representation*.[4] Note that now [19]

$$v_l = \dim(R_l)\,. \tag{3.3}$$

Taking the dimension of $\mathbb{C}^\alpha\Gamma$ as a vector space over $\mathbb{C}$, this implies a result which will be useful later on:

$$|\Gamma| = \sum_l v_l^2\,. \tag{3.4}$$

We may now prove the following theorems:

[4]One may extend the use of language to call $\mathbb{C}^\alpha\Gamma$ the regular representation even when α is nontrivial. This was implicit in [7].

JHEP02(2001)009

Theorem 1. *If $R = \mathbb{C}^{\alpha}\Gamma$ and Γ acts freely on $\mathbb{C}^n$ outside the origin then $\mathscr{M}_R$ is the orbifold $\mathbb{C}^n/\Gamma$ itself.*

Theorem 2. *If $R = \mathbb{C}^{\alpha}\Gamma$ and Γ does not act freely on $\mathbb{C}^n$ outside the origin then $\mathscr{M}_R$ consists of more than one component, one of which is the orbifold $\mathbb{C}^n/\Gamma$ itself.*

To do this we follow the proof of theorem 4.2 in [13] with a little modification to allow for discrete torsion.

First let A_i denote the operator $\mathrm{ad}(X_i)$, that is $\mathsf{A}_i(X) = [X_i, X]$. We also have the Hermitian conjugate operator $\mathsf{A}_i^{\dagger}$. The equations (3.1) and (3.2) together with the Jacobi identity show that

$$\sum_i \mathsf{A}_i^{\dagger}\mathsf{A}_i(X_j^{\dagger}) = 0\,. \tag{3.5}$$

It is a basic fact of linear algebra that $\mathsf{A}_i^{\dagger}\mathsf{A}_i$ is a positive operator. Therefore the above implies that $\mathsf{A}_i^{\dagger}\mathsf{A}_i(X_j^{\dagger}) = 0$ for any choice of i or j. That is, $\mathsf{A}_i(X_j^{\dagger}) = 0$ for any choice of i or j. In particular

$$\mathsf{A}_i(X_i^{\dagger}) = [X_i^{\dagger}, X_i] = 0\,. \tag{3.6}$$

This means that each matrix X_i is *normal* and can be diagonalized by conjugation by a unitary matrix $U \in \mathscr{G} = \mathrm{U}(m)$. Furthermore (3.1) implies that X_i can be simultaneously diagonalized for all i.

Let $v_1 \in R$ be a simultaneous eigenvector of X_i with eigenvalues $\lambda_1^{(i)}$. Denote the basis of $Q = \mathbb{C}^n$ by q_i, $i = 1, \ldots, n$. We then write

$$\lambda_1 = \sum_i \lambda_1^{(i)} q_i \in Q\,. \tag{3.7}$$

Now, the fact that $X \in (Q \otimes \mathrm{End}\, R)^{\Gamma}$ tells us that

$$\sum_j Q_{ij}(\gamma)R(\gamma)X_j = X_i R(\gamma) \tag{3.8}$$

and so

$$\begin{aligned}\sum_j Q_{ij}(\gamma)R(\gamma)X_j v_1 &= X_i R(\gamma) v_1 \\ &= \sum_j Q_{ij}(\gamma)\lambda_1^{(j)} R(\gamma) v_1.\end{aligned} \tag{3.9}$$

In other words $R(\gamma)v_1$ is an eigenvector of X with eigenvalue $Q(\gamma)\lambda_1$. *Note in particular that the eigenvalues must always appear as Γ-orbits in Q.* Let us denote $R(\gamma)v_1$ by v_γ.

Now let us first assume that the action of Γ on Q has no fixed points away from the origin. Then assuming v_1 is non-zero, the eigenvalues are all distinct and so the vectors v_γ will form an orthogonal basis for $R = \mathbb{C}^\alpha\Gamma$. We may fix this basis to be orthonormal.

Let us choose a fixed orthonormal basis for $\mathbb{C}^\alpha\Gamma$ by fixing e_1 and then defining $e_\gamma = \gamma e_1$. The unitary matrix U diagonalizing the X_i can be taken to rotate the v_γ basis into the e_γ basis. That is, $Uv_\gamma = e_\gamma$, for all γ. Note that for any $\mu \in \Gamma$,

$$\begin{aligned} R(\mu)Uv_\gamma &= R(\mu)e_\gamma \\ &= \mu e_\gamma \\ &= \alpha(\mu,\gamma)e_{\mu\gamma}\,, \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} UR(\mu)v_\gamma &= UR(\mu)R(\gamma)v_1 \\ &= \alpha(\mu,\gamma)UR(\mu\gamma)v_1 \\ &= \alpha(\mu,\gamma)Uv_{\mu\gamma} \\ &= \alpha(\mu,\gamma)e_{\mu\gamma}\,. \end{aligned} \tag{3.11}$$

Thus U and $R(\mu)$ commute which implies that $U \in \mathscr{G}^\Gamma$.

So now, we see that a point $\mathscr{M}_R$ is determined by the eigenvalues λ, up to permutations. What we have shown above is that this set is precisely in one-to-one correspondence with Γ-orbits in Q. That the correspondence is onto is easy to see by taking diagonal matrices. This proves theorem 1.

We now come to the more interesting case of a non-isolated fixed point at the origin. Again the analysis of Sardo Infirri [13] applies even with discrete torsion switched on.

A fairly simple argument in linear algebra may be used to show that $Q/\Gamma \subset \mathscr{M}_R$. Let x be any point in Q and let Γ_x be the subgroup of Γ which fixes x. We assume Γ_x is trivial for a generic $x \in Q$. Given x, we may construct X such that its eigenvalues are given by the Γ-orbit of x with each eigenvalue appearing with multiplicity $|\Gamma_x|$. Now this matrix will have eigenvectors forming an orthonormal basis of R and we again recover the above construction. Note also that if x lies in an open neighbourhood such that Γ acts freely on this neighbourhood then this inclusion $Q/\Gamma \hookrightarrow \mathscr{M}_R$ is locally a homeomorphism on this subset. Thus Q/Γ appears as a component of $\mathscr{M}_R$.

A key point however is that if Γ_x is nontrivial we may choose X to have eigenvalues x of multiplicity *less than* $|\Gamma_x|$. Suppose we have a point $x \in Q$ away from the origin which is fixed by $\Gamma_x \subsetneq \Gamma$. The Γ-orbit of x will now have fewer than $|\Gamma|$ points. We may build an X which is diagonal and whose eigenvalues correspond to the Γ-orbit of x together with zero for the remaining eigenvalues. This is clearly a valid X which lies outside any one-to-one correspondence between $\mathscr{M}_R$ and Q/Γ. Thus we see that theorem 1 *must* fail if the quotient singularity is not isolated at the origin. This proves theorem 2.

3.2 Mobile wrapped branes

These extra branches of $\mathscr{M}_R$ correspond to the wrapped branes discussed in [15]. We have shown that one must always obtain extra branches of the moduli space corresponding to wrapped branes if the quotient singularity in Q/Γ is not isolated.

The analysis of [13] tells us exactly how to describe these extra branches. Again the description remains valid with discrete torsion switched on. Suppose x is a point away from the origin fixed by Γ_x as above. We may consider an α-twisted projective representation R_x of Γ_x. This "induces" a representation of Γ of the form

$$\mathrm{Ind}_{\Gamma_x}^{\Gamma} R_x = \mathbb{C}^\alpha\Gamma \otimes_{\mathbb{C}^\alpha\Gamma_x} R_x\,, \tag{3.12}$$

(where $\mathbb{C}^\alpha\Gamma$ is viewed as a left $\mathbb{C}^\alpha\Gamma$-module and a right $\mathbb{C}^\alpha\Gamma_x$-module). This induced representation will represent a "wrapped brane" which can be constrained to live along the fixed locus of Γ_x if R_x is chosen to have sufficiently low dimension. Using an irreducible representation for R_x will always yield such an example. If we take the "regular representation" $\mathbb{C}^\alpha\Gamma_x$ for R_x then clearly

$$\begin{aligned}\mathrm{Ind}_{\Gamma_x}^{\Gamma} R_x &= \mathbb{C}^\alpha\Gamma \otimes_{\mathbb{C}^\alpha\Gamma_x} \mathbb{C}^\alpha\Gamma_x \\ &= \mathbb{C}^\alpha\Gamma.\end{aligned} \tag{3.13}$$

This shows how a suitable sum of these wrapped branes "stuck" along the fixed point set will combine to give the brane associated to $\mathbb{C}^\alpha\Gamma$. This represents the set of wrapped branes corresponding to one or more of the extra branches of the moduli space. Looking at all fixed subspaces in the orbifold will account for all the branches.

Conversely, given a representation R of Γ, we may describe the moduli space associated to R as follows. R can be induced by various subgroups. We need to consider each *minimal* such subgroup, in the sense that it contains no other subgroups that induce R. The moduli space corresponding to R will be the union of the fixed point sets of these minimal subgroups.

We will give several examples of these induced representations in section 5.

4. The McKay correspondence

4.1 An attempt with homology

The general idea of the McKay correspondence is as follows. Let $Y \to \mathbb{C}^n/\Gamma$ be a crepent (i.e., maintaining the Calabi-Yau condition) maximal blow-up. The homology of Y is then closely related to the representation theory of Γ.

One of the most straight-forward but crude ways of seeing this correspondence in D-branes arises from the work of [2, 5]. Let us begin by considering the particular case of the representation $R = \mathbb{C}^\alpha\Gamma$ as in the previous section. Each node in the quiver associated to a D-brane on a orbifold can be associated to a potential Fayet-Iliopoulos term in the D-brane action. It is well-established (see [13, 14, 5, 20] for example) that adding such terms into the action can result in a blow-up of $\mathscr{M}_R = \mathbb{C}^n/\Gamma$. Note that this fact remains true even when discrete torsion is switched on.[5] The way to picture these blow-ups is most easily seen in terms of *moment maps*. Note that

$$\rho = \sum_{i=1}^{n} [X_i^\dagger, X_i]\,, \tag{4.1}$$

represents a moment map $\rho : X \to \mathrm{Lie}^* \mathscr{G}^\Gamma$. The effect of adding Fayet-Iliopoulos terms to the action is to effectively shift this moment map $\rho \to \rho - \zeta$ for some new moment map ζ lying in the centre of $\mathrm{Lie}^* \mathscr{G}^\Gamma$. We refer to [13] for more details. The centre of $\mathscr{G}^\Gamma$ corresponds to U(1)'s associated to each vertex of the quiver. We have fixed our basis of representations R_l to label each of these vertices. Thus we may expand

$$\zeta = \sum_p \zeta_p R_p^*\,, \tag{4.2}$$

where R_p^* is a basis *dual* to the representations R_l.

One may add the same terms to the action by adding insertions of closed string twist fields. Let us review the case for no discrete torsion. An appendix in [2] showed how these twist fields are related precisely to the Fayet-Iliopoulos terms. Let ϕ_γ be a field in the NS-NS sector of the string theory twisted by γ. Then $\zeta_p = \sum_\gamma \mathrm{tr}\, R_p^*(\gamma)\phi_\gamma$, where the sum is taken over the conjugacy classes of Γ. Using the orthogonality of the characters $\chi_p^*(\gamma) = \mathrm{tr}\, R_p^*(\gamma)$ we may rewrite this as

$$\phi_\gamma = \sum_p \chi_p(\gamma)\zeta_p\,, \tag{4.3}$$

(where we have rescaled ϕ_γ).

Including the possibility of discrete torsion actually makes little difference to this argument. It was argued in [8, 21, 9] that the twisted strings in a theory with discrete torsion are precisely those corresponding to α-regular conjugacy classes. Now the theory of characters of projective representations and α-regular conjugacy classes is essentially identical to the usual theory of characters and conjugacy classes. We refer to [19] for more details. In fact, equation (4.3) is still true with discrete torsion switched on so long as we use the relevant notions of projective representations.

For a given $\gamma \in \Gamma$, we may write the eigenvalues of γ as $\exp(2\pi i a_1)$, $\exp(2\pi i a_2)$, $\ldots$, $\exp(2\pi i a_n)$ where $0 \le a_i < 1$. It is then a well-known result of topological field theory that we may then associate ϕ_γ with an element of $H^{2w(\gamma)}$ with

$$w(\gamma) = \sum_i a_i\,. \tag{4.4}$$

[5] The choice of discrete torsion considered in most examples in the literature results in a system so tightly constrained that no blow-ups remain. A sufficiently general example will still have blow-ups however.

JHEP02(2001)009

(Presumably this cohomology group relates to cohomology with compact support as our target space is not compact.) Notable fields are those obeying $w(\gamma)=1$. These correspond to marginal deformations and relate to deformations of the Kähler form in H^2. The formalism of topological field theory extends this to the case $w(\gamma)\neq 1$.

To return to the basis R_l naturally associated to the vertices of our quiver we need to take the dual of this mapping and so we relate the group generated by the representations of Γ to the *homology* of the resolution. We may then state our first draft of the D-brane McKay correspondence:

Conjecture 1. *Let Y be a maximal crepent resolution of $\mathbb{C}^n/\Gamma$ allowed by a given choice (possibly trivial) of discrete torsion. For any $\gamma\in\Gamma$ we associate a cycle in $H_{2w(\gamma)}(Y)$ to the D-brane with representation $\sum_l \chi_l^*(\gamma)R_l$.*

One can have immediate success with this conjecture by considering the case of the untwisted sector given by $\gamma=1$. This conjecture then says that the associated homology class lies in H_0 and corresponds to $\sum_l \dim(R_l)R_l=\mathbb{C}^\alpha\Gamma$. In section 3 we saw how the representation $\mathbb{C}^\alpha\Gamma$ had the entire target space $\mathbb{C}^n/\Gamma$ as (at least one component) its moduli space. This agrees beautifully with it being the moduli space of a single point. This idea of associating a point with $\mathbb{C}^\alpha\Gamma$ indeed goes back to the original work of [2].

It is not hard to see from arguments along the lines of section 3 that taking R to be two copies of $\mathbb{C}^\alpha\Gamma$ will give a symmetric product of $\mathbb{C}^n/\Gamma$ with itself (again with potentially further branches which we ignore for now). This is thus the moduli space of two points. Clearly this argument works for any number of points. We will therefore take this conjecture to be true in the case of H_0. That is, *a point is always given by the representation $\mathbb{C}^\alpha\Gamma$.*

As soon as one tries to go beyond this simple case one immediately runs into difficulty. The reason for this is clear. In the language of [22, 23], the twisted fields ϕ_γ are a natural basis of cohomology in the *orbifold* phase whereas one would normally like to picture the homology in question as living in the large radius *Calabi-Yau* phase. One therefore needs to change basis between these phases to get the right statement for the McKay correspondence. This change of basis was essentially described in [24] and has been described in exactly this context in [6]. We refer to these references for more details.

Having said all this, it might come as a surprise that the case of H_0 worked out so nicely. Why didn't we have to mix in other twist fields to to get the classical H_0? A special rôle is given to H_0 essentially because we are working in a *noncompact* example of $\mathbb{C}^n/\Gamma$. We will not attempt a general proof here but it can be seen from the examples of [24, 6] that in this case one has a simple constant as one possible solution of the Picard-Fuchs equation. This is not the case for a compact Calabi-Yau manifold and so one assumes that in such a case H_0 would mix freely with the other dimensions.

JHEP02(2001)009

Even after allowing for this mixing within the homology, our conjecture is not really satisfactory. In particular, considering wrapped branes will not give us information on torsion classes in the homology. We need to discuss K-theory to improve matters.

4.2 K-theory

One of the most general forms of the McKay correspondence was given in [25] which states the McKay correspondence in terms of derived categories of coherent sheaves. This in turn implies a statement about K-theory. Although it is clear that derived categories should play an important rôle in string theory we will restrict ourselves to the weaker language of K-theory in this paper.

As is well-known, K-theory is similar to (singular) cohomology but can differ. In the context of D-branes it is now well-established that K-theory is the more relevant notion [26, 27]. It is not surprising therefore that D-branes give a nice picture of the McKay correspondence when the language of K-theory is used.

Consider a type-IIB string theory on the orbifold $\mathbb{C}^n/\Gamma$ with a choice of discrete torsion α. Let D_0 denote the lattice of resulting D-brane charges. Crudely stated, we know that D_0 is associated to $H^{\mathrm{even}}(\mathbb{C}^n/\Gamma, \mathbb{Z})$ in some way.

Let us assert our version of the McKay correspondence:

Proposition 2. *The lattice D_0 is isomorphic to the free abelian group generated by the α-twisted irreducible projective representations of Γ.*

When $\alpha = 1$ this reduces to the usual McKay correspondence and the relationship between D-branes and K-theory. For nontrivial α it is probably difficult to prove this proposition given the current status of our definitions of string theory in a singular space. Indeed one may wish to regard this proposition as a definition of D_0.

In order to get a better feeling for this conjecture we need to relate things to our discussion of homology above. As we saw, it really is homology rather than cohomology which is most naturally associated to the representation theory of Γ. Because of this, let us introduce a group K_0 which is a homological version of K-theory. It is usual to define K-theory as an abstract cohomology theory in the Eilenberg-Steenrod sense but one may also define the corresponding theory of abstract homology.

One may proceed as follows. Define $K_{\mathrm{even}}(\mathrm{point}) = \mathbb{Z}$ and $K_{\mathrm{odd}}(\mathrm{point}) = 0$. Now define K_* as a theory of homology in the Eilenberg-Steenrod sense (see, for example, of [28, section 2.3]). This is sufficient to define K_{even} and K_{odd} for any topological space. Replacing homology by cohomology here would give the usual K-theory.

For a fairly large class of toplogical spaces one can relate K_0 to the usual *singular* homology H_*. This may be done by using the Atiyah-Hirzebruch spectral sequence [29] rewritten in terms of homology. This has also been used recently in the physics literature in [30]. We simply quote the following

Theorem 3. *Let Y be a finite simplicial complex. There exists a homology spectral sequence with*

$$E^2_{p,q} = H_p(Y, K_q(\text{point}))\,, \tag{4.5}$$

which converges to give $E^\infty_{p,q} \cong K_{p+q}(Y)$.

This theorem encodes two ways in which $K_0(Y)$ may differ from $H_{\text{even}}(Y,\mathbb{Z})$. Firstly one may have nontrivial differentials $\partial_r : E^r_{p,q} \to E^r_{p-r,q+r-1}$. The effect of this is to kill elements of $K_0(Y)$ which appear in $H_{\text{even}}(Y,\mathbb{Z})$. While this certainly can happen, such effects tend not to occur until one considers homology of a large dimension. We are mainly concerned with 0-cycles and 2-cycles in this paper and *we will ignore such a possibility.* We thus assume that $E^2_{p,q} = E^\infty_{p,q}$.

Of more interest is the fact that $E^\infty_{p,q}$ is associated to a filtration. Ignoring the effect of the differentials, there is a sequence of inclusions[6]

$$K_0 = K_0^\infty \supset \cdots \supset K_0^4 \supset K_0^2 \supset K_0^0\,, \tag{4.6}$$

where

$$H_n \cong \frac{K_0^n}{K_0^{n-2}}\,, \tag{4.7}$$

and $K_0^{-2} = 0$.

Returning to the world of string theory, we know that singular (co)homology is *not* the relevant notion for describing string states on an orbifold with, or without, torsion. One needs to use "stringy (co)homology". We will assert that the group D_0 essentially plays the rôle an object like K_0 except that it is related to stringy homology via the Atiyah-Hirzebruch spectral sequence. *From now on we will use the symbol K_0 to refer to this stringy object.* Note that K_0 knows about discrete torsion. This is similar to the approach of [9]. Preliminary aspects of the mathematics of this object have been conjectured [31]. We use equation (4.7) as our *definition* of stringy homology.

Note that (4.7) is not quite the same thing as saying

$$K_0 = \bigoplus_{i \text{ even}} H_i\,, \tag{4.8}$$

as we will see shortly. Indeed (4.8) is false in general. Having said that, (4.8) is correct if one considers *rational* (or real or complex) coefficients rather than integers. In this way we really should see a McKay correspondence for homology from D-branes if we ignore considerations such as torsion.

Finally in this section we would like to prove the following:

[6]Note this this filtration is in the opposite direction to the usual K-theory associated with cohomology.

Theorem 4. *The stringy homology of $\mathbb{C}^n/\Gamma$ contains a torsion group of order p as a subgroup, where p is the order of the discrete torsion α in the group $H^2(\Gamma, \mathrm{U}(1))$. Furthermore, the group H_2 is torsion-free if α is trivial.*

Suppose first that α is trivial. Note that K_0 and thus all the subgroups K_0^n are free. Next note that if α is trivial then one of the representations of Γ is the trivial representation, R_0, which is one dimensional. Now the regular representation $\mathbb{C}\Gamma$ breaks up into a sum $\oplus_l \dim(R_l)R_l$ and so R_0 appears with multiplicity one. Since $\mathbb{C}\Gamma$ generates K_0^0 we may construct $H_2 = K_0^2/K_0^0$ from K_0^2 simply by eliminating R_0 as a generator. Thus H_2 is free and we prove the second part of the theorem.

The proof of the first part of the theorem requires the introduction of some more technicalities and the reader may wish to skip the rest of this section. The "representation group", $\hat{\Gamma}$, of a group Γ over the field $\mathbb{C}$ is defined as follows. Suppose we have the following central extension of Γ:

$$1 \to A \xrightarrow{i} \hat{\Gamma} \overset{\overset{s}{\curvearrowleft}}{\xrightarrow{j}} \Gamma \to 1\,, \tag{4.9}$$

where A is a finite group isomorphic to $H^2(\Gamma, \mathrm{U}(1))$. The map s is a set-theoretic map acting as a right-inverse to j. Given any linear representation of $\hat{\Gamma}$ we may use s to define a projective representation of Γ, The group $\hat{\Gamma}$ is said to be a representation group for Γ over $\mathbb{C}$ if any projective representation may be lifted to a linear representation of $\hat{\Gamma}$. It is established (see 3.3 in [19]) that $\hat{\Gamma}$ exists for any Γ.

Let R be an N-dimensional irreducible projective representation of Γ twisted by a specific $\alpha \in H^2(\Gamma, \mathrm{U}(1))$. We lift this to an N-dimensional irreducible linear representation $\hat{R}$ of $\hat{\Gamma}$. Since $i(A)$ is central, the representation in $\hat{R}$ of any element of $i(A)$ must be proportional to the identity matrix. There is natural isomorphism $\mathrm{Hom}(A, \mathrm{U}(1)) \to H^2(\Gamma, \mathrm{U}(1))$ which explicitly gives $\hat{R}(i(A))$ as a subgroup of $\mathrm{U}(1)$ for a given choice of α.

Consider the projective representation ring of Γ. If R_1 is an α_1-twisted projective representation of Γ and R_2 is an α_1-twisted projective representation then $R_1 \otimes R_2$ is an $(\alpha_1 + \alpha_2)$-twisted projective representation where we use the natural group structure of $H^2(\Gamma, \mathrm{U}(1))$. Thus if α is of order p in $H^2(\Gamma, \mathrm{U}(1))$, then the elements of $\hat{R}(i(A))$ must be of order p. Indeed it must be that $\hat{R}(i(A)) \cong \mathbb{Z}_p$.

Now note that one cannot have a one-dimensional truly projective representation. This is because the choice of α in such a representation would necessarily be the coboundary of a one-cocycle. Taking determinants of the matrices $\hat{R}$ above, gives a one-dimensional representation which is an submodule of the tensor product $R^{\otimes N}$. This implies that the Nth power of any element of A must be trivial in the representation $\hat{R}(i(A))$. *This implies that N is a multiple of p.*

The above argument shows that all the irreducible projective representations of Γ have dimension equal to a multiple of some integer p given by the order of α.

This implies that the representation $\mathbb{C}^\alpha\Gamma = \oplus_l \dim(R_l)R_l$ is p-divisible. Since K_0^0 is generated by $\mathbb{C}^\alpha\Gamma$, this means that K_0/K_0^0 has an element of p-torsion. This proves the first part of the theorem.

Note that we have not proven that this torsion subgroup lives entirely in H_2. The form of the filtration could conceivably force it to be spread over many homology groups. It would also be interesting if we could prove that p is the greatest common divisor of the dimensions of the projective representations of Γ. This would show that the torsion in K_0/K_0^0 is precisely $\mathbb{Z}_p$. We will not attempt this here.

5. Examples

5.1 $\mathbb{Z}_2 \times \mathbb{Z}_2$

We first consider everyone's favorite example of discrete torsion. Let Γ be the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ acting on $\mathbb{C}^3$ generated by $a = \mathrm{diag}(-1,-1,1)$ and $b = \mathrm{diag}(1,-1,-1)$. It is well-known that $H^2(\mathbb{Z}_2\times\mathbb{Z}_2,\mathrm{U}(1)) \cong \mathbb{Z}_2$. Hence there are two choices — one may have discrete torsion switched on or off. This example gives rise to a non-isolated singularity. There are three complex lines of $\mathbb{Z}_2$-fixed points passing through the origin.

	R_0	R_1	R_2	R_3
1	1	1	1	1
a	1	-1	1	-1
b	1	1	-1	-1
ab	1	-1	-1	1

Table 1: The irreducible representations of $\mathbb{Z}_2 \times \mathbb{Z}_2$.

5.1.1 No discrete torsion

Let us first discuss the case without discrete torsion. Many aspects of this have been analyzed in the context of D-branes in [32] for example. Since this is an abelian group, there are as many representations as there are group elements — i.e., four. Each representation is one-dimensional and is listed in table 1. We show the quiver in figure 1. Note that if we have an arrow going from vertex i to vertex j and another arrow from vertex j back to vertex i then we combine them into a single line in this diagram. The simplifies the appearance of the diagrams in this paper.[7]

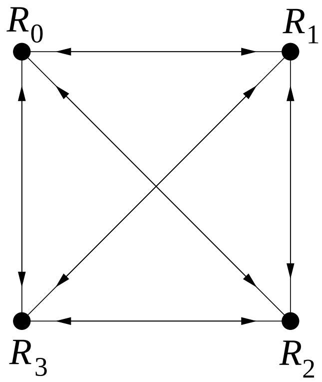

Figure 1: The quiver for $\mathbb{Z}_2 \times \mathbb{Z}_2$ with no discrete torsion.

We see therefore that K_0 is the free abelian group generated by four elements which we call $k_0, \ldots, k_3$. The regular representation is given by $\mathbb{C}\Gamma = R_0 \oplus R_1 \oplus R_2 \oplus R_3$ and thus $k_0 + k_1 + k_2 + k_3$ corresponds to a point in $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$.

Computing the value of $w(\gamma)$ from (4.4) for each of the group elements shows that we expect H_0 to be dimension one and H_2 to be dimension three with H_n dimension zero for $n > 2$. Thus the filtration associated to the Atiyah-Hirzebruch

[7] Indeed all the arrows appearing in this paper have this property. This is certainly not true in general.

spectral sequence implies that

$$H_2 \cong \frac{K_0}{K_0^0} \cong \frac{K_0}{H_0}. \tag{5.1}$$

That is, we may consider $H_2 \cong \mathbb{Z}^3$ to be generated by k_1, k_2, k_3 with a redundant generator $k_0 = -k_1 - k_2 - k_3$.

In order to map this to the classical picture for the homology of this orbifold, we need to blow-up. The toric picture for a blow-up is given in figure 2. We refer to [33] for example for more details.

This resolution contains three isolated rational curves C_1, C_2, C_3 shown as lines in figure 2. We claim that these curves correspond to the generators k_1, k_2, k_3.

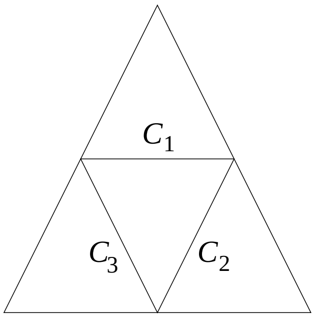

Figure 2: A blow-up of $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$.

This would mean that C_1 is represented by the representation R_1. That is, the integers v_l of section 2 are given by $(v_0, v_1, v_2, v_3) = (0, 1, 0, 0)$. Constructing the moduli space for such a quiver is rather trivial and $\mathscr{M}_{R_1}$ is a point. This is consistent with this curve being *isolated.*

Now consider the homology class $C_1 + C_3$. The geometry of the blow-up dictates that this is the class of a 2-cycle that is free to move along the resolution of a fixed *line* of the orbifold action. Indeed, for $(v_0, v_1, v_2, v_3) = (0, 1, 0, 1)$ one may show that $\mathscr{M}_{R_1 \oplus R_3}$ is given by $\mathbb{C}/\mathbb{Z}_2$.

We now show this is consistent with the discussion at the end of section 3. Let $\Gamma_x \cong \mathbb{Z}_2$ be the subgroup of $\mathbb{Z}_2 \times \mathbb{Z}_2$ generated by a. This fixes a complex line. Let R_x be trivial representation of this $\mathbb{Z}_2$ subgroup and let R'_x be the induced representation of Γ. Now $R'_x = \mathbb{C}^\Gamma \otimes_{\mathbb{C}^\Gamma_x} R_x$ is easily seen to be two-dimensional and generated by $1 \otimes 1$ and $b \otimes 1$. With respect to this basis one computes 1 and a to be given by the identity matrix; and b and ab is given by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

This implies that $R'_x = R_0 \oplus R_2$. Similarly if R_x were the nontrivial representation of $\mathbb{Z}_2$ then $R'_x = R_1 \oplus R_3$. Thus the regular representation can be decomposed $\mathbb{C}\Gamma = (R_0 \oplus R_2) \oplus (R_1 \oplus R_3)$ to give two wrapped branes running up and down the lines fixed by a. Note that in homology $k_0 + k_2 = -(k_1 + k_3)$ and so these two branes are oppositely oriented consistent with the description in [15].

The K-theory story gives the full picture. The representations $R_0 \oplus R_2$ and $R_1 \oplus R_3$ both correspond to the homology element given by the two-cycle $C_1 + C_3$. The sum of these representations gives a point (rather than a trivial cycle).

In the case of the regular representation we have various components for $\mathscr{M}_{R_0 \oplus R_1 \oplus R_2 \oplus R_3}$:

1. A point may move anywhere in $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$. This gives a component of the moduli space equal to $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$.

2. Two wrapped branes may move up and down the complex line fixed by a. This gives a component equal to $\mathbb{C}^2/(\mathbb{Z}_2)^3$.

3. Two wrapped branes may move up and down the complex line fixed by b. This gives another component equal to $\mathbb{C}^2/(\mathbb{Z}_2)^3$.

4. Two wrapped branes may move up and down the complex line fixed by ab. This gives another component equal to $\mathbb{C}^2/(\mathbb{Z}_2)^3$.

5.1.2 With discrete torsion

Now let us switch discrete torsion on. The discrete quaternion group H can be written as a central extension:

$$1 \to \mathbb{Z}_2 \to \mathsf{H} \to \mathbb{Z}_2 \times \mathbb{Z}_2\,. \tag{5.2}$$

This defines a cocycle in $H^2(\mathbb{Z}_2\times\mathbb{Z}_2, \mathbb{Z}_2)$ which defines a cocycle $\alpha \in H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathrm{U}(1))$. The cocycle is given in table 2.[8] This case was analyzed in [7].

		γ_2: 1	a	b	ab
γ_1	1	1	1	1	1
	a	1	-1	1	-1
	b	1	-1	-1	1
	ab	1	1	-1	-1

Table 2: The value of $\alpha(\gamma_1, \gamma_2)$.

The proof of theorem 4 shows that any irreducible representation of Γ must have a dimension which is a multiple of two. Since Γ has only four elements, (3.4) implies that there can only be a single irreducible representation, R_0, of dimension two. The rather trivial quiver in this case is shown in figure 3.

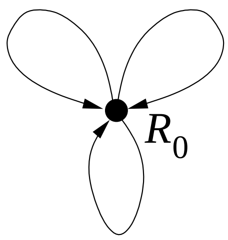

Figure 3: The quiver for $\mathbb{Z}_2 \times \mathbb{Z}_2$ with discrete torsion.

The only α-regular conjugacy class is the identity. The value of $w(1)$ is zero and so we have H_0 having rank one. There are no twisted strings states which generate H_n for $n > 0$. However, it is not correct to say that H_0 is the only nontrivial stringy homology group.

It is clear that $\mathbb{C}^\alpha\Gamma = 2R_0$. This implies that $K_0/K_0^0 \cong \mathbb{Z}_2$ which means that *one of the stringy homology groups H_{2a} is given by $\mathbb{Z}_2$ for some $a \geq 1$*. It is not clear how this torsion cycle could be declared to be a 2-cycle rather than a 4-cycle etc. All that we see is that there is a torsion cycle somewhere!

For the wrapped branes, let us consider a subset $\mathbb{Z}_2 \subset \Gamma$ which fixes some complex line. Restricting to this subgroup, α-twisted projective representations become linear representations. Either of the irreducible representations of $\mathbb{Z}_2$ induce the projective representation R_0 of Γ. Thus R_0 gives the brane stuck along the fixed lines.

[8] It is *not* possible to write a valid cocycle in any way as $\alpha(a^m b^n, a^{m'} b^{n'}) = i^{mn'-m'n}$ as has been claimed in the literature.

Again in the case of the representation $\mathbb{C}^{\alpha}\Gamma$ we have various components for $\mathscr{M}_{2R_0}$:[9]

1. A point may move anywhere in $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$. This gives a component of the moduli space equal to $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$.

2. Two wrapped branes may move up and down the complex line fixed by a. This gives a component equal to $\mathbb{C}^2/(\mathbb{Z}_2)^3$.

3. Two wrapped branes may move up and down the complex line fixed by b. This gives another component equal to $\mathbb{C}^2/(\mathbb{Z}_2)^3$.

4. Two wrapped branes may move up and down the complex line fixed by ab. This gives another component equal to $\mathbb{C}^2/(\mathbb{Z}_2)^3$.

5. One wrapped brane may move along the line fixed by a and one wrapped brane may move along the line fixed by b. This gives another component equal to $\mathbb{C}^2/(\mathbb{Z}_2)^3$.

6. One wrapped brane may move along the line fixed by a and one wrapped brane may move along the line fixed by ab. This gives another component equal to $\mathbb{C}^2/(\mathbb{Z}_2)^3$.

7. One wrapped brane may move along the line fixed by b and one wrapped brane may move along the line fixed by ab. This gives another component equal to $\mathbb{C}^2/(\mathbb{Z}_2)^3$.

In particular, there are a good deal more components than there were in the case with no discrete torsion. This is because the two wrapped branes are not now paired. One may obtain the K-theory element corresponding to a point by adding *any* two wrapped branes together. They need not originate from the same fixed line.

5.2 A trihedral group

Let a and b generate $\mathbb{Z}_2 \times \mathbb{Z}_2$ as above. Let

$$g = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{5.3}$$

In this section we will let Γ be the twelve element group generated by a, b and g. This is a "trihedral group" and is probably the easiest nonabelian to analyze in the context of the McKay correspondence. It has been analyzed in [34, 35, 20] for example.

[9]The extra components were missed in the analysis of [7].

Note that $g^{-1}ag = b$, $g^{-1}bg = ab$, and $g^{-1}abg = a$. This gives a $\mathbb{Z}_3$ action on $\mathbb{Z}_2 \times \mathbb{Z}_2$ and realizes Γ as the semi-direct product $\mathbb{Z}_3 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$. One may now use the Hochschild-Serre spectral sequence (see section VII.6 of [36] for example) to show that $H^2(\Gamma, \mathrm{U}(1)) \cong \mathbb{Z}_2$. Thus, as in the previous example, we have a choice of no discrete torsion or a unique nontrivial discrete torsion.

5.2.1 No discrete torsion

It is a simple matter to determine that Γ has 4 conjugacy classes given by $\{1\}$, $\{a, b, ab\}$, $\{g, ag, bg, abg\}$ and $\{g^2, ag^2, bg^2, abg^2\}$. The characters are shown in table 3, where $\omega = \exp(2\pi i/3)$. R_0 is the trivial representation and R_3 is the same as Q. The quiver is shown in figure 4.

Now $w(1) = 0$ and $w(a) = w(g) = w(g^2) = 1$. This leads to $H_0 \cong \mathbb{Z}$ and $H_2 \cong \mathbb{Z}^3$.

	R_0	R_1	R_2	R_3
1	1	1	1	3
a, b, ab	1	1	1	-1
g, ag, bg, abg	1	ω	ω^2	0
g^2, ag^2, bg^2, abg^2	1	ω^2	ω	0

Table 3: The characters of $\mathbb{Z}_3 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$.

Next note that the $\mathbb{Z}_3$ subgroup of Γ generated by g fixes a line. Let us find the representations of Γ corresponding to wrapped branes running along this line. A little group theory shows that the trivial representation of $\mathbb{Z}_3$ induces the representation $R_0 \oplus R_3$ of Γ while the other two irreducible representations of $\mathbb{Z}_3$ induce the representations $R_1 \oplus R_3$ and $R_2 \oplus R_3$. The representation of a point which is $\mathbb{C}\Gamma = R_0 \oplus R_1 \oplus R_2 \oplus 3R_3$ may thus break up into these three wrapped branes in accord with the usual picture.

We also have a line of fixed points generated by the $\mathbb{Z}_2$ subgroup generated by a. (This is identified with the lines fixed by b and ab by the $\mathbb{Z}_3$ action.) Again, it is an exercise in group theory to show that the trivial representation of $\mathbb{Z}_2$ induces the representation $R_0 \oplus R_1 \oplus R_2 \oplus R_3$. The nontrivial irreducible representation of $\mathbb{Z}_2$ induces the representation $2R_3$. Again these two representations add up to give the regular representation.

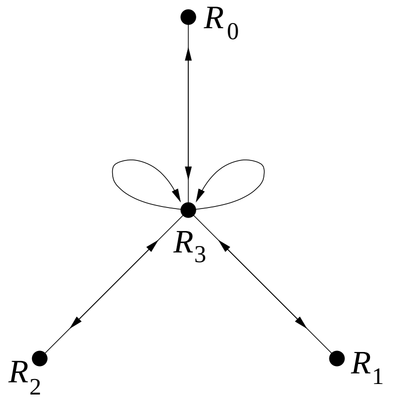

Figure 4: The quiver for $\mathbb{Z}_3 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$ with no discrete torsion.

Note that for the $\mathbb{Z}_2$-fixed line there is a definite lack of symmetry between the two wrapped branes. One of them comes from a representation which is 2-divisible. That is, the state corresponding to the representation R_3 is stuck at the origin where as twice this representation gives a state that is free to run along the $\mathbb{Z}_2$ fixed line. This is a little reminiscent of torsion even though there there is no torsion in this model.

5.2.2 With discrete torsion

The quaternion group H admits a $\mathbb{Z}_3$ automorphism leading to the group $\mathbb{Z}_3 \ltimes \mathsf{H}$. This can be written as a central extension

$$1 \to \mathbb{Z}_2 \to \mathbb{Z}_3 \ltimes \mathsf{H} \to \Gamma \to 1\,, \tag{5.4}$$

where Γ is our desired trihedral group. This central extension gives an explicit representation of the group cocycle corresponding to the nontrivial choice of discrete torsion.

	R_0	R_1	R_2
1	2	2	2
$g,\ ag,\ bg,\ abg$	-1	$-\omega$	$-\omega^2$
$g^2,\ ag^2,\ bg^2,\ abg^2$	-1	$-\omega^2$	$-\omega$

Table 4: The projective characters of $\mathbb{Z}_3 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$.

The analysis of section 4.2 together with equation (5.4) tells us that every projective representation of Γ must have an even dimension. Since Γ has twelve elements, (3.4) implies that there must be three irreducible projective representations each of dimension two. The three α-regular conjugacy classes are $\{1\}$, $\{g, ag, bg, abg\}$ and $\{g^2, ag^2, bg^2, abg^2\}$. The character table is given in table 4 and the quiver in figure 5.

The fields twisted by g and g^2 both have $w = 1$. Thus we appear to have two generators for H_2. The representation $\mathbb{C}^\alpha\Gamma$ is 2-divisible and so we have 2-torsion somewhere in H_*. Either one writes $H_2 \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2$, or the 2-torsion goes into a higher H_n. As in the previous case with discrete torsion, it is difficult to ascribe any dimensionality to this torsion cycle.

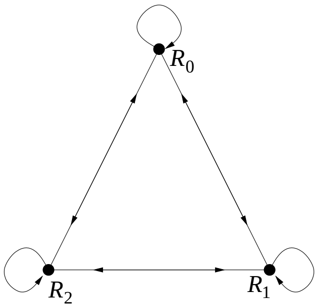

Figure 5: The quiver for $\mathbb{Z}_3 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$ with discrete torsion.

The element g fixes a complex line as before. This $\mathbb{Z}_3$ subgroup's three irreducible representations induce $R_1 \oplus R_2$, $R_0 \oplus R_2$ and $R_0 \oplus R_1$ respectively as the projective representations of Γ. These are our three wrapped branes that move on this complex line.

The element a fixes another complex line as before. The two irreducible representations of the corresponding $\mathbb{Z}_2$ both induce the projective representation $R_1 \oplus R_2 \oplus R_3$. This representation is "half of a point" and gives a state that is free to move along this fixed line. Twice this state can move anywhere.

6. Discussion of strings versus D-branes

We would like to draw attention to the fact that theorem 4 does *not* say that the torsion in the homology of an orbifold with discrete torsion is given by $H^2(\Gamma, \mathrm{U}(1))$. Rather we say that α may take on any value of $H^2(\Gamma, \mathrm{U}(1))$ and, for a given choice, the homology of the orbifold will naturally contain a cyclic torsion component $\mathbb{Z}_p$ where p depends upon α.

The examples in section 5 both had $H^2(\Gamma, \mathrm{U}(1)) \cong \mathbb{Z}_2$ and the nontrivial choice of α yielded $p = 2$. Life can get more complicated than this however. It is quite possible that $H^2(\Gamma, \mathrm{U}(1))$ is not a cyclic group. For $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \subset \mathrm{SU}(4)$, one has $H^2(\Gamma, \mathrm{U}(1)) \cong (\mathbb{Z}_2)^3$ for example.[10] The torsion in the homology we found in section 4 is cyclic and can *never* therefore equal the full discrete torsion group in this case. (We should point out that there may be other contributions to torsion in the homology groups of dimension > 2 but this is not the torsion in homology we are naturally associating to the discrete torsion.)

This, combined with the K-theory nature of D-brane charge gives a definite asymmetry between the string and the D1-brane as we now discuss.

Consider a type-IIB string on a space Y. The B-field which is associated to the string charge is valued in $H^2(Y, \mathrm{U}(1))$. It generally believed that the B field therefore encodes the discrete torsion degree of freedom. This has been established rigorously by Sharpe [37, 38, 39] if one views the B-field as a gerbe connection.

The analogue degree of freedom for the D1-brane comes from the RR degrees of freedom which live on a torus associated to the lattice of D-brane charges. In the case of $Y = \mathbb{C}^n/\Gamma$ we have proposition 2 which implies that this torus has dimension given by the number of irreducible (projective) representations of Γ. Two points are worth noting:

1. This torus is a connected space — there are never discrete degrees of freedom. This is because the lattice of charges in proposition 2 is a free group.

2. The dimensionality depends upon a choice α of discrete torsion.

Clearly this implies that there is no analogue of "discrete torsion" in the RR sector. The B-field degrees of freedom and the RR degrees of freedom are completely different. We believe that the interpretation of this fact is that *one cannot truly claim that there is an S-duality of the type-IIB string which exchanges the string with the D1-brane.* Note that S-duality was also analyzed in a related context in [30]. It would be interesting to understand the relation between their work and ours.

Note that this is consistent with the point of view that such a duality is likely to be killed if we look at vacua with too little supersymmetry [40, 41, 42]. In order to obtain nontrivial discrete torsion we must compactify on a space with holonomy at least SU(3).[11] Thus we only see these peculiarities in the discrete degrees of freedom for theories with eight supercharges or fewer.

Note finally that for examples where the orbifold action is free, one might expect discrete degrees of freedom in the RR sector. An example of this was analyzed

[10] It would be interesting to see if any finite subgroup of SU(3) gives a noncyclic discrete torsion group.

[11] For any finite group $\Gamma \subset \mathrm{SU}(2)$ one may show that $H^2(\Gamma, \mathrm{U}(1))$ is trivial. This may be done by applying the Cartan-Leray spectral sequence to the free quotient S^3/Γ.

in [43, 44] where are definite choice of an apparent discrete degree of freedom was required in order to obtain "black hole level matching". Clearly we do not yet completely understand the RR degrees of freedom in general.

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